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An Instructive Counterexample to a Maximality Theorem of Raúl Fierro

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ABSTRACT
We intensively investigate a very particular situation when

$$X = \{ x \in \mathbb{R}^2 : \quad x_1 + x_2 \leq 0 \};$$

$$\varphi(x) = x_1 + x_2, \qquad f(x) = (x_1 + 1, \ x_2 - 1);$$

$$S(x) = \{ y \in X : \quad \varphi(x) \leq \varphi(y) \}$$

for all $x \in X$.

This example shows, in particular, that an implication stated in a maximality theorem, published by Raúl Fierro in 2017, is not true without assuming the anti-symmetry of the corresponding preorder.

A true particular case of this theorem improves and supplements a former similar theorem of Sehie Park from 2000, and has to be proved just after Zorn's lemma and a maximality principle of H. Brézis and F. Browder.

KEYWORDS
Preorder relations, maximal elements, fixed points

1. Introduction

In [17], by using a preorder $\preceq$ on a nonempty set X and the notation

$$S(x, \preceq) = \{ y \in X : \quad x \preceq y \},$$

Raúl Fierro tried to prove the following partial improvement of [35, Theorem 1] of Sehie Park without mentioning his former similar papers [32, 33].

Theorem 1.1. *Let $x_0 \in X$. The following eight conditions are equivalent:* **(1)** ***there exists a maximal element $x^* \in X$ such that $x_0 \preceq x^*$;***

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(2) ***there exists*** $x_1 \in S(x_0, \preceq)$ ***such that for each chain*** C ***in*** $S(x_1, \preceq)$***,*** $\bigcap_{x \in C} S(x, \preceq) \neq \emptyset$***;***

(3) ***there exist*** $x_1 \in S(x_0, \preceq)$ ***and a maximal chain*** C^* ***in*** $S(x_1, \preceq)$ ***such that*** $\bigcap_{x \in C^*} S(x, \preceq) \neq \emptyset$***;***

(4) ***for each*** $T : S(x_0, \preceq) \to 2^X$ ***such that, for each*** $x \in S(x_0, \preceq) \setminus Tx$***, there exists*** $y \in X \setminus \{x\}$ ***satisfying*** $x \preceq y$***, there exists*** $z \in S(x_0, \preceq)$ ***such that*** $z \in Tz$***;***

(5) ***any function*** $f : S(x_0, \preceq) \to X$ ***such that*** $x \preceq f(x)$***, for all*** $x \in S(x_0, \preceq)$***, has a fixed point;***

(6) ***for each*** $T : S(x_0, \preceq) \to 2^X \setminus \{\emptyset\}$ ***such that*** $x \preceq y$***, for all*** $x \in S(x_0, \preceq)$ ***and*** $y \in Tx$***, there exists*** $z \in S(x_0, \preceq)$ ***such that*** $Tz = \{z\}$***;***

(7) ***any family*** $\mathcal{F}$ ***of functions*** $f : S(x_0, \preceq) \to X$ ***such that*** $x \preceq f(x)$***, for all*** $x \in S(x_0, \preceq)$***, has a common fixed point;***

(8) ***for any subset*** Y ***of*** X ***such that*** $S(x_0, \preceq) \cap Y = \emptyset$***, there exists*** x ***in*** $S(x_0, \preceq) \setminus Y$ ***satisfying*** $S(x, \preceq) = \{x\}$***.***

In the present paper, we shall intensively investigate the interesting particular case when

$$X = \{ x \in \mathbb{R}^2 : \quad x_1 + x_2 \leq 0 \};$$

$$\varphi(x) = x_1 + x_2, \qquad\qquad f(x) = (x_1 + 1, \; x_2 - 1);$$

$$S(x) = \{ y \in X : \quad \varphi(x) \leq \varphi(y) \}$$

for all $x \in X$***.***

This example will show, in particular, that the implication (3) $\Longrightarrow$ ***(4) in Theorem 1.1 is not true without assuming the antisymmetry of the relation*** $\preceq$***.***

Namely, in this case, it can be easily seen that $X(S) = (X, S)$ ***is a nonvoid, totally preordered, but not partially ordered set.***

Moreover, under the notations

$$x_0 = x_1 = 0 = (0, 0) \qquad \textit{and} \qquad C^* = S(0) = \varphi^{-1}(0),$$

we have $x_0 \in X$***,*** $x_1 \in S(x_0)$***, and*** C^* ***is a maximal chain in*** $S(x_1)$***.***

Furthermore, by using our forthcoming Theorem 6.2, we can see that

$$\bigcap_{x \in C^*} S(x) = \bigcap_{x \in S(0)} S(x) \supseteq \bigcap_{x \in X} S(x) = S(0) \supseteq \{0\} \neq \emptyset.$$

Therefore, the corresponding particular case of assertion (3) holds.

On the other hand, by defining

$$T(x) = \{ f(x) \} \quad \textbf{\textit{for all}} \qquad x \in X,$$

we can easily see that, for each $x \in S(x_0) \setminus T(x)$, ***we have*** $f(x) \in X \setminus \{x\}$ ***such that*** $f(x) \in S(x)$.

However, despite this, for each $z \in X$ ***we have*** $z \notin T(x)$. ***Therefore, the corresponding particular case of assertion (4) does not hold.***

A relational improvement of a true particular case of Theorem 1.1 has been proved in [7], where the curious assertion (8) has also been reformulated.

This improvement generalizes and supplements a former similar theorem of Park [35, Theorem 1]. (See also [32, 33] for some more general settings.)

Moreover, it has to be treated just after the famous Zorn lemma [30, p. 532], and a useful maximality principle of Brézis and Browder [8, Corollary 2].

2. A Few Basic Facts on Relations

A subset R ***of a product set*** $X \times Y$ ***is called a*** **relation on** X **to** Y. ***In particular, a relation*** R ***on*** X ***to itself is called a*** **relation on** X. ***And,*** $\Delta_X = \{(x, x) : \ x \in X\}$ ***and*** X^2 ***are called the*** **identity** ***and*** **universal relations** ***on*** X, ***respectively.***

If R ***is a relation on*** X ***to*** Y, ***then for any*** $x \in X$ ***and*** $A \subseteq X$ ***the sets*** $R(x) = \{y \in Y : \ (x, y) \in R\}$ ***and*** $R[A] = \bigcup_{a \in A} R(a)$ ***are called the*** **images** ***or*** **neighbourhoods** ***of*** x ***and*** A ***under*** R, ***respectively.***

If $(x, y) \in R$, ***then instead of*** $y \in R(x)$, ***we may also write*** $x\,R\,y$. ***However, instead of*** $R[A]$ ***we cannot write*** $R(A)$. ***Namely, it may occur that, in addition to*** $A \subseteq X$, ***we also have*** $A \in X$.

Now, the sets $D_R = \{ x \in X : \ R(x) \neq \emptyset \}$ ***and*** $R[X]$ ***may be called the*** **domain** ***and*** **range** ***of*** R, ***respectively. And, if*** $D_R = X$, ***then we may say that*** R ***is a*** **relation of** X **to** Y, ***or that*** R ***is a*** **non-partial relation on** X **to** Y.

If R ***is a relation on*** X ***to*** Y ***and*** $E \subseteq D_R$, ***then the relation*** $R\,|\,E = R \cap (E \times Y)$ ***is called the*** **restriction** ***of*** R ***to*** E. ***Moreover, if*** R ***and*** S ***are relations on*** X ***to*** Y ***such that*** $D_R \subseteq D_S$ ***and*** $R = S\,|\,D_R$, ***then*** S ***is called an*** **extension** ***of*** R.

In particular, a relation f ***on*** X ***to*** Y ***is called a*** **function** ***if for each*** $x \in D_f$ ***there exists*** $y \in Y$ ***such that*** $f(x) = \{y\}$. ***In this case, by identifying singletons with their elements, we may simply write*** $f(x) = y$ ***instead of*** $f(x) = \{y\}$.

Moreover, a function $\star$ ***of*** X ***to itself is called a*** **unary operation** ***on*** X. ***While, a function*** $*$ ***of*** X^2 ***to*** X ***is called a*** **binary operation** ***on*** X. ***And, for any*** $x, y \in X$, ***we usually write*** $x^\star$ ***and*** $x * y$ ***instead of*** $\star(x)$ ***and*** $*(x, y)$, ***respectively.***

If R is a relation on X to Y, then a function f of D_R to Y is called a **selection function** *of R if $f(x) \in R(x)$ for all $x \in D_R$. Thus, by the Axiom of Choice [26], we can see that every relation is the union of its selection functions.*

For a relation R on X to Y, we may naturally define two **set-valued functions** *φ_R of X to $\mathcal{P}(Y)$ and Φ_R of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_R(x) = R(x)$ for all $x \in X$ and $\Phi_R(A) = R[A]$ for all $A \subseteq X$.*

Functions of X to $\mathcal{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful objects than relations on X to Y. In [66], they were briefly called **corelations on X to Y**.

However, if U is a relation on $\mathcal{P}(X)$ to Y and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that U is a **super relation** *and V is a* **hyper relation on X to Y** *[42, 69]. Thus, closures (proximities) [72] are super (hyper) relations.*

Note that a super relation on X to Y is an arbitrary subset of $\mathcal{P}(X) \times Y$. While, a corelation on X to Y is a particular subset of $\mathcal{P}(X) \times \mathcal{P}(Y)$. Thus, set inclusion is a natural partial order for super relations, but not for corelations.

For a relation R on X to Y, the relation, $R^c = (X \times Y) \setminus R$ is called the **complement** *of R. Thus, it can be shown that $R^c(x) = R(x)^c = Y \setminus R(x)$ for all $x \in X$, and $R^c[A]^c = \bigcap_{a \in A} R(a)$ for all $A \subseteq X$.*

Moreover, the relation $R^{-1} = \{(y, x) : \ (x, y) \in R\}$ is called the **inverse** *of R. Thus, it can be shown that $R^{-1}[B] = \{x \in X : \ R(x) \cap B \neq \emptyset\}$ for all $B \subseteq Y$, and in particular $D_R = R^{-1}[Y]$.*

If R is a relation on X to Y, then we have $R = \bigcup_{x \in X} \{x\} \times R(x)$. Therefore, the values $R(x)$, where $x \in X$, uniquely determine R. Thus, a relation R on X to Y can also be naturally defined by specifying $R(x)$ for all $x \in X$.

For instance, if S is a relation on Y to Z, then the **composition** *$S \circ R$ can be defined such that $(S \circ R)(x) = S[R(x)]$ for all $x \in X$. This is equivalent to the* **box product** *[62], which can be directly defined for any family of relations.*

3. Some Important Relational Properties

Now, a relation R on X, i. e., a subset R of X^2, may be briefly defined to be **reflexive** *and* **transitive** *if under the plausible notations $R^0 = \Delta_X$ and $R^2 = R \circ R$, we have $R^0 \subseteq R$ and $R^2 \subseteq R$, respectively.*

Moreover, R may be briefly defined to be **symmetric** *and* **antisymmetric** *if $R^{-1} \subseteq R$ and $R \cap R^{-1} \subseteq R^0$, respectively. And, R may be briefly defined to be* **total** *and* **directive** *if $X^2 \subseteq R \cup R^{-1}$ and $X^2 \subseteq R^{-1} \circ R$, respectively.*

In the sequel, as it is usual, a reflexive and transitive (symmetric) relation will be called a **preorder (tolerance) relation**. *And, a symmetric (antisymmetric) preorder relation will be called an* **equivalence (partial order) relation**.

For a relation R on X, by using $R^0 = \Delta_X$, we may also define $R^n =$

$R \circ R^{n-1}$ ***for*** $n \in \mathbb{N}$ ***. Moreover, we may also define*** $R^{\infty} = \bigcup_{n=0}^{\infty} R^n$ ***. Thus,*** R^{∞} ***is the smallest preorder relation on*** X ***containing*** R ***[20].***

For a set A ***and an increasing sequence*** $\mathcal{A} = (A_n)_{n=1}^{\infty}$ ***in*** $\mathcal{P}(X)$ ***, the relations*** $R_A = A^2 \cup (A^c \times X)$ ***and*** $R_{\mathcal{A}} = \Delta_X \cup \bigcup_{n=1}^{\infty} (A_n \times A_n^c)$ ***may be called the Pervin and the Cantor preorders on*** X ***[34, 41], respectively.***

Note that if R ***is only reflexive relation on*** X ***and*** $x \in X$ ***, then*** $\mathcal{A}_R(x) = (R^n(x))_{n=1}^{\infty}$ ***is already an increasing sequence in*** $\mathcal{P}(X)$ ***. Thus, the preorder relation*** $R_{\mathcal{A}_R(x)}$ ***may also be naturally considered.***

Moreover, for a real function φ ***of*** X ***and a quasi-pseudo-metric*** d ***on*** X ***[18, p. 3], the relation*** $R_{(\varphi,d)} = \{(x, y) \in X^2 : \quad d(x, y) \leq \varphi(y) - \varphi(x)\}$ ***may be called the Brøndsted or the Bishop-Phelps preorder on*** X ***[9].***

From $R_{(\varphi,d)}$ ***, by letting*** φ ***and*** d ***to be the zero functions, we can obtain the specialization and preference preorders*** $R_d = \{(x, y) \in X^2 : \quad d(x, y) = 0\}$ ***and*** $R_{\varphi} = \{(x, y) \in X^2 : \quad \varphi(x) \leq \varphi(y)\}$ ***on*** X ***[15, 73], respectively.***

In this respect, it is also worth mentioning that the **divisibility relation** ***on*** $\mathbb{Z}$ ***, the*** **subsequence relation** ***on*** $X^{\mathbb{N}}$ ***, and the*** **refines** ***and*** **divides relations** ***for covers, relations and relators are also, in general, only preorder relations [50].***

Besides the usual basic properties, several further remarkable relational properties were studied in [51]. For instance, a relation R ***on*** X ***was called*** **quasi-anti-symmetric** ***if*** $y \in R(x)$ ***and*** $x \in R(y)$ ***imply*** $R(x) = R(y)$ ***for all*** $x, y \in X$ ***.***

Much more importantly, a relation R ***on*** X ***was called*** **non-mingled-valued** ***if*** $R(x) \cap R(y) \neq \emptyset$ ***implies*** $R(x) = R(y)$ ***for all*** $x, y \in X$ ***. Thus, for instance, it can be shown that all equivalence and linear relations [56] are non-mingled-valued.***

The latter two properties, by using the reasonable notations $R^{-} = R^{-1} \circ R$ ***and*** $R^{\circ} = \left(R^{-1} \circ R^c\right)^c$ ***, can be reformulated in the forms*** $R \cap R^{-1} \subseteq R^{\circ}$ ***and*** $R \circ R^{-} \subseteq R$ ***, respectively. Note that if*** R ***is non-partial, then*** $R \subseteq R \circ R^{-}$ ***holds.***

4. Generalized Ordered Sets

If R ***is a relation on*** X ***, then analogously to the widely used abbreviation*** **poset** ***of Birkhoff [3] the ordered pair*** $X(R) = (X, R)$ ***may be called a*** **goset** ***(generalized ordered set) [64], instead of a*** **relational system** ***[4, 43].***

If P ***is a relational property, then the goset*** $X(R)$ ***will be said to have property*** P ***if the relation*** R ***has this property. For instance, the goset*** $X(R)$ ***will be called*** **reflexive** ***if*** R ***is a reflexive relation on*** X ***.***

In particular, the goset $X(R)$ ***will be called a*** **proset** ***(preordered set) if*** R ***is a preorder on*** X ***[64]. Moreover, the abbreviations*** **toset** ***(totally ordered set) and*** **woset** ***(well-ordered set) of Rudeanu [44] can also be well used.***

Thus, every set X ***is a poset with the identity relation*** Δ_X ***. Moreover,*** X ***is a proset with the universal relation*** X^2 ***. And, the power set*** $\mathcal{P}(X) = \{A : \quad A \subseteq X\}$ ***of*** X ***is a poset with the ordinary set inclusion*** $\subseteq$ ***.***

If R ***is a relation on*** X ***to*** Y***, then the ordered pair*** $(X, Y)(R) = \big((X, Y), R\big)$ ***is called a*** **formal context** ***[16, 19]. Instead of "formal context", the terms "relational space" or "simple relator space" may also be well used.***

Namely, if $\mathcal{R}$ ***is a family of relations on*** X ***to*** Y***, then the ordered pair*** $(X, Y)(\mathcal{R})$
$= \big((X, Y), \mathcal{R}\big)$ ***was called a*** **relator space** ***[55]. Thus, relator spaces are substantial generalizations of several algebraic and topological structures.***

Some important notions used in posets, metric and topological spaces can be naturally generalized to relator spaces [49]. However, instead of arbitrary relators it is frequently enough to work only with preorder relations and relators [1, 58].

If R ***is a relation on*** X ***to*** Y***, then for any*** $B \subseteq Y$ ***we may naturally define*** $\operatorname{Int}_R(B) = \{A \subseteq X : \ R[A] \subseteq B\}$***, and*** $\operatorname{int}_R(B) = \{x \in X : \ \{x\} \in \operatorname{Int}_R(B)\}$***. Moreover, we may also naturally define*** $\mathcal{E}_R = \{ B \subseteq Y : \ \operatorname{int}_R(B) \neq \emptyset \}$***.***

Furthermore, we may also naturally define $\operatorname{Lb}_R(B) = \{ A \subseteq X : \ A \times B \subseteq R \}$***, and*** $\operatorname{lb}_R(B) = \{x \in X : \ \{x\} \in \operatorname{Lb}_R(B)\}$***. However, these tools are not independent from the former ones. Namely, by [55], we have*** $\operatorname{Lb}_R = \operatorname{Int}_{R^c} \circ \mathcal{C}_Y$***.***

Now, if R ***is a relation on*** X***, then for any*** $A \subseteq X$ ***we may also naturally define*** $\min_R(A) = A \cap \operatorname{lb}_R(A)$ ***and*** $\sup_R(A) = \min_R\big(\operatorname{ub}_R(A)\big)$***, where*** $\operatorname{ub}_R = \operatorname{lb}_{R^{-1}}$***. Thus, if*** R ***is antisymmetric, then*** $\min_R(A)$ ***and*** $\sup_R(A)$ ***are at most singletons.***

If A ***is a subset of a goset*** $X(R)$***, then*** $A(R \cap A^2)$ ***is also a goset. Therefore, for instance, the set*** A ***may be naturally called total if this subgoset is total. That is,*** $R \cap A^2$ ***is a total relation on*** A***.***

Now, by a usual terminology, a total subset A ***of a goset*** $X(R)$ ***may be briefly called a*** **chain**. ***Thus, by an extension of the Hausdorff Maximal Principle [30], we can state that each nonvoid proset contains, at least one, maximal chain.***

An element x ***of a goset*** $X(R)$ ***will be called*** **maximal** ***if*** $x R y$ ***implies*** $y R x$ ***for all*** $y \in X$***. That is,*** R ***is symmetric at*** x ***in the sense that*** $R(x) \subseteq R^{-1}(x)$***. Thus,*** R ***is symmetric if and only if every element of*** $X(R)$ ***is maximal.***

Moreover, the element x ***will be called*** **strongly maximal** ***if*** $x R y$ ***implies*** $y = x$ ***for all*** $y \in X$***. That is,*** R ***is dominated by*** Δ_X ***at*** x ***in the sense that*** $R(x) \subseteq \Delta_X(x)$***. Thus, if*** R ***is reflexive at*** x***, then the corresponding equality also holds.***

Note that if in particular $x \in \max_R(X)$***, then*** x ***is a maximal element of*** $X(R)$***. However, the converse statement need not be true. Moreover, if*** R ***is reflexive and antisymmetric, then "maximal" and "strongly maximal" are equivalent notions.***

5. Further Important Relational Properties

A function f ***of one goset*** $X(R)$ ***to another*** $Y(S)$ ***may be naturally called increasing if*** $u\,R\,v$ ***implies*** $f(u)S\,f(v)$ ***for all*** $u, v \in X$. ***And,*** f ***may be called*** **decreasing** ***if it is increasing as a function of*** $X(R)$ ***to the dual*** $Y(S^{-1})$ ***of*** $Y(S)$.

Moreover, the function f ***may be called*** **strictly increasing** ***if*** $u\,R\,v$ ***and*** $u \neq v$ ***imply*** $f(u)\,S\,f(v)$ ***and*** $f(u) \neq f(v)$ ***for all*** $u, v \in X$. ***However, to define a strict form of*** R, ***instead of*** $R \setminus \Delta_X$, ***the relation*** $R \setminus R^{-1}$ ***can also be used [40].***

Now, instead of "increasing" we may also naturally write "continuous". Namely, f ***is increasing if and only if*** $v \in R(u)$ ***implies*** $f(v) \in S\big(f(u)\big)$. ***That is, if*** v ***is in the*** R***–neighbourhood of*** u, ***then*** $f(v)$ ***is in the*** S***–neighbourhood of*** $f(u)$.

This property can be reformulated in several forms. For instance, it can be proved that f ***is increasing if and only if one of the inclusions*** $f \circ R \subseteq S \circ f$, $R \subseteq f^{-1} \circ S \circ f$, $f \circ R \circ f^{-1} \subseteq S$ ***and*** $R \circ f^{-1} \subseteq f^{-1} \circ S$ ***holds [67].***

Now, a relation F ***on a goset*** $X(R)$ ***to a set*** Y ***may be naturally called*** **inclusion-increasing** ***if the associated set-valued function*** φ_F ***is increasing. That is,*** $u\,R\,v$ ***implies*** $F(u) \subseteq F(v)$ ***for all*** $u, v \in X$.

However, if F ***is a relation on*** $X(R)$ ***to*** $Y(S)$, ***then in addition to the above inclusion-increasingness of*** F, ***we may also define an*** **order-increasingness** ***of*** F ***by requiring the implication*** $u \in \mathrm{lb}_R(v) \implies F(u) \in \mathrm{Lb}_S\big(F(v)\big)$ ***for all*** $u, v \in X$.

Thus, it can be shown that F ***is inclusion-increasing if and only if*** $R \circ F^{-1} \subseteq F^{-1}$, ***or equivalently*** F^{-1} ***is ascending-valued . And,*** F ***is order-increasing if and only if*** $F^{-1} \circ R \circ F^{-1} \subseteq S$, ***or equivalently*** $F[\,R(u)] \subseteq \mathrm{ub}_S\big(F(u)\big)$ ***for all*** $u \in X$ ***[67].***

If $+$ ***is a binary operation on a set*** X, ***then the ordered pair*** $X(+) = (X, +)$ ***will be called an*** **additive** ***(or additively written)*** **groupoid**. ***Though, the term "groupoid" is recently rather used for a "Brandt groupoid" [13, p. 99].***

In this case, for any $x \in X$ ***and*** $A, B \subseteq X$, ***we may also naturally define*** $x + A = \{x + a : \ a \in A\}$ ***and*** $A + B = \{\,a + b : \ a \in A, \ b \in B\,\}$. ***Thus,*** $\mathcal{P}(X)$ ***is also a groupoid which may be considered as an extension of*** X.

According to [52, 54, 59], a relation R ***on a groupoid*** X ***will be called a*** **translation relation** ***if*** $x + R(y) \subseteq R(x+y)$ ***for all*** $x, y \in X$. ***Actually, in this case,*** R ***should be rather called a left-super-translational relation on*** X.

However, if R ***is such a relation on a group*** X, ***then we can already prove that*** $R(x) = x + R(0)$, ***and thus*** $R(x+y) = x + R(y)$ ***for all*** $x, y \in X$. ***Moreover, if*** X ***is commutative, then we can also note that*** $R(x) + y = x + R(y)$.

The latter equality does not require a translation property of the domain of R. ***Thus,*** **partial multipliers** ***(the multiplicative forms of partial translation functions) can be well used for the extensions of commutative structures [45, 46, 53].***

Quite similarly, a relation F on one groupoid X to another Y could be naturally called additive if $F(x)+F(y) \subseteq F(x+y)$ for all $x, y \in X$. However, in this case, we shall rather say that F is a **super additive relation** ***on X to Y.***

Therefore, in this paper, the relation F will be called **additive** ***[70] if it is not only superadditive, but also*** **subadditive** ***in the sense that $F(x+y) \subseteq F(x)+F(y)$ for all $x, y \in X$. That is, the corresponding equality holds for all $x, y \in X$.***

If in particular X is a vector space over $\mathbb{R}$, then for any $\lambda \in \mathbb{R}$ and $A \subseteq X$, we may also naturally define $\lambda A = \{\lambda a : \ a \in A\}$. Thus, A may be briefly defined to be a **cone** ***if $\lambda A \subseteq A$ for all $\lambda > 0$. Moreover, A will be called*** **pointed** ***if $0 \in A$.***

Thus, a subset A of X is a pointed cone if and only if it is nonvoid and $\lambda A \subseteq A$ for all $\lambda \in \mathbb{R}_+ = [0, +\infty[$. Moreover, the cone A will usually be assumed to be **convex** ***in the usual sense that $\lambda A + (1-\lambda)A \subseteq A$ for all $\lambda \in [0, 1]$.***

Thus, a subset A of X is a pointed convex cone if and only if it is nonvoid and $\lambda A + \mu A \subseteq A$ for all $\lambda, \mu \in \mathbb{R}_+$. To derive this, note that if $\lambda + \mu \neq 0$, then $\lambda(\lambda+\mu)^{-1}A + \mu(\lambda+\mu)^{-1}A \subseteq A$ and thus $\lambda A + \mu A \subseteq (\lambda+\mu)A \subseteq A$.

Now, for instance, a relation F on a cone A in a vector space X to another vector space Y may be naturally called **λ–superhomogeneous***, for some $\lambda > 0$ if $\lambda F(x) \subseteq F(\lambda x)$ for all $x \in X$.*

Moreover, F may be called **positively superhomogeneous** ***if it is λ–superhomogeneous for all $\lambda > 0$. Thus, by using that $\lambda^{-1}F(\lambda x) \subseteq F\big(\lambda^{-1}(\lambda x)\big) = F(x)$, and thus $F(\lambda x) \subseteq \lambda F(x)$, we can see that F is actually*** **positively homogeneous.**

6. An Instructive Example

Notation 6.1. ***Define***

(1) $X = \{ x \in \mathbb{R}^2 : \ x_1 + x_2 \leq 0 \}$;

(2) $\varphi(x) = x_1 + x_2$ ***for all*** $x \in X$;

(3) $f(x) = (x_1 + 1, \ x_2 - 1)$ ***for all*** $x \in X$;

(4) $S(x) = \{ y \in X : \ \varphi(x) \leq \varphi(y) \}$ ***for all*** $x \in X$.

Remark 6.1. *Hence, by writing $\preceq$ instead of S, we can see that*

$$x \preceq y \iff x\,S\,y \iff y \in S(x) \iff \varphi(x) \leq \varphi(y)$$

for all $x, y \in X$. Thus, in particular φ is increasing *with respect to $\preceq$ and $\leq$.*

Remark 6.2. *Moreover, by writing T instead of $\leq$, we can also easily see that*

$$y \in S(x) \iff \varphi(x) \leq \varphi(y) \iff \varphi(x)\,T\,\varphi(y) \iff \varphi(y) \in T\big(\varphi(x)\big)$$
$$\iff y \in \varphi^{-1}\big[T\big(\varphi(x)\big)\big] \iff y \in \big(\varphi^{-1} \circ T \circ \varphi\big)(x)$$

for all $x, y \in X$. *Therefore,* $S(x) = \big(\varphi^{-1} \circ T \circ \varphi\big)(x)$ *for all* $x \in X$, *and thus*

$$S = \varphi^{-1} \circ T \circ \varphi .$$

That is, S *is* projectively generated *from* T *by* φ *[48].*

Now, by using definitions (1) – (4), we can also easily prove the following two theorems.

Theorem 6.1. *The following assertions hold:*

(1) $0 \in X$ ***and*** $\varphi(0) = 0$ ***;*** **(2)** $\varphi = \varphi \circ f$ ***;***

(3) $\varphi(x) = \langle x, (1, 1) \rangle$ ***;*** **(4)** $f(x) = x + (1, -1)$ ***for all*** $x \in X$***.***

Proof. To prove (2), note that

$$\varphi(x) = x_1 + x_2 = x_1 + 1 + x_2 - 1 = \varphi\big(x_1 + 1, \, x_2 - 1\big) = \varphi\big(f(x)\big) = (\varphi \circ f)(x)$$

for all $x \in X$. □

Remark 6.3. *From assertions (3) and (4), we can see that* φ *is additive and non-negatively homogeneous, and* f *is a translation function on* X.

Theorem 6.2. *The following assertions hold:*

(1) S ***is total preorder on*** X***;*** **(2)** S ***is not antisymmetric;***

(3) $S(0) = \varphi^{-1}(0)$ ***;*** **(4)** $S(0) = \{(r, -r)\}_{r \in \mathbb{R}}$ ***;***

(5) $S(0) = \bigcap_{x \in X} S(x)$ ***;*** **(6)** $S(0) = \{y \in X : \; X = S^{-1}(y)\}$***;***

(7) $0 \in S(x)$ ***for all*** $x \in X$ ***;*** **(8)** $y \in S(x)$ ***for all*** $x, y \in S(0)$ ***.***

Proof. By the definition of S, it is clear that (1) is true. Moreover, if for instance $x = (-1, 0)$ and $y = (0, -1)$, then we can see that $x_1 + x_2 = -1$ and $y_1 + y_2 = -1$. Thus, $x, y \in X$ such that $\varphi(x) \le \varphi(y)$ and $\varphi(y) \le \varphi(x)$. Therefore, $y \in S(x)$ and $x \in S(y)$, despite that $x \ne y$. Thus, (2) is also true.

On the other hand, for any $y \in \mathbb{R}^2$, we have

$$y \in S(0) \iff \varphi(y) \le 0, \quad \varphi(0) \le \varphi(y) \iff \varphi(y) = 0 \iff y \in \varphi^{-1}(0).$$

Therefore, (3) is true. Moreover, now we can also note that

$$y \in \varphi^{-1}(0) \iff \varphi(y) = 0 \iff y_1 + y_2 = 0 \iff y_2 = -y_1 .$$

Therefore, $\varphi^{-1}(0) = \{(r, -r) : \; r \in \mathbb{R}\}$, and thus (4) is also true.

Now, for any $y \in X$, we can also note that

$$y \in S(0) \iff \varphi(y) = 0 \iff \forall \; x \in X : \; \varphi(x) \le \varphi(y) \iff \forall \; x \in X : \; y \in S(x) \iff y \in \bigcap_{x \in X} S(x).$$

Therefore, (5) is also true. Moreover, we can also note that

$$y \in S(0) \iff \forall\ x \in X: \ \ y \in S(x) \iff \forall\ x \in X: \ \ x \in S^{-1}(y) \iff S^{-1}(y) = X.$$

Therefore, (6) is also true.

Finally, if $x \in X$, then by (1) and (5) we can see that $0 \in S(0) \subseteq S(x)$, and thus (7) holds. Moreover, if $x, y \in S(0)$, then by (3) we have $\varphi(x) = 0$ and $\varphi(y) = 0$. Therefore, $\varphi(x) \leq \varphi(y)$, and thus $y \in S(x)$ trivially holds. Thus, (8) also hold. □

Remark 6.4. *From the totality of S, we can see that every subset A of X is total with respect to the relation S. That is, A is a chain in $X(S)$.*

Moreover, because of the reflexivity and transitivity of S, for any $x, y \in X$ we have

$$x\,S\,y \iff y \in S(x) \iff S(y) \subseteq S[\,S(x)\,] \iff S(y) \subseteq S(x).$$

Therefore, the inclusion properties of the values $S(x)$ uniquely determine S. Moreover, the relation S is increasing *with respect to itself and $\subseteq$.*

Remark 6.5. *From assertion (1), we can also see that $S \cap S^{-1}$ is the* largest symmetric (equivalence) *relation contained in S.*

Thus, in particular, the identity function Δ_X of X is selection function *of not only of S, but also of $S \cap S^{-1}$.*

Remark 6.6. *Assertions (7) and (8) can be reformulated in the forms that:*

(1) $S^{-1}(0) = X$; *(2)* $S(0)^2 \subseteq S$.

Remark 6.7. *By using the corresponding definitions, we can see that*

$$\min\big((\varphi \circ S)(x)\big) = \min\big(\varphi[\,S(x)\,]\big) = \min_{y \in S(x)} \varphi(y) = \varphi(x)$$

and

$$\max\big((\varphi \circ S)(x) = \max\big(\varphi[\,S(x)\,]\big) = \max_{y \in S(x)} \varphi(y) = \varphi(0) = 0$$

for all $x \in X$. Thus, in particular we can also state that

$$\rho(x) = \sup\big((\varphi \circ S)(x)\big) = 0$$

for all $x \in X$.

Remark 6.8. *While, from assertion (8), we can see that the restriction of S to $S(0)$ is just the* universal relation *on $S(0)$.*

Moreover, we can also note hat $\{0\}$ is the largest antisymmetric subset *of $S(0)$.*

7. Two Further Theorems on the Relation S

Since $\varphi(x) \leq 0$ ***for all*** $x \in X$, ***we may also naturally introduce the following***

Notation 7.1. For any $x \in X$, we define

$$\Phi(x) = [\,\varphi(x),\ 0\,].$$

Namely, by using this notation, in addition to Remark 6.2, we can also easily establish the following instructive reformulation of the definition of the relation S **.**

Theorem 7.1. *We have*

$$S = \varphi^{-1} \circ \Phi.$$

Proof. By the corresponding definitions, for any $x, y \in X$, we have

$$\begin{aligned} y \in S(x) \quad &\Longleftrightarrow \quad \varphi(x) \leq \varphi(y) \leq 0 \quad \Longleftrightarrow \quad \varphi(y) \in\ ]-\infty,\ 0] \cap [\,\varphi(x),\ +\infty[\\ &\Longleftrightarrow \quad \varphi(y) \in [\,\varphi(x),\ 0\,] \quad \Longleftrightarrow \quad \varphi(y) \in \Phi(x) \quad \Longleftrightarrow \quad y \in \varphi^{-1}[\,\Phi(x)\,]. \end{aligned}$$

Therefore,

$$S(x) = \varphi^{-1}[\,\Phi(x)\,] = \big(\varphi^{-1} \circ \Phi\big)(x)$$

for all $x \in X$, and thus the required equality is also true. □

Remark 7.1. *From the above theorem, we can also see that*

$$\big(\varphi^{-1} \circ \varphi\big)(x) = \varphi^{-1}\big(\varphi(x)\big) \subseteq \varphi^{-1}\big([\,\varphi(x),\ 0\,]\big) = \varphi^{-1}[\,\Phi(x)\,] = S(x)$$

for all $x \in X$, *and thus* $\varphi^{-1} \circ \varphi \subseteq S$.

However, the latter inclusion is even more obvious from Remark 6.2 by the reflexivity of T . *Note that, by Remark 6.2, we could use the more simple, but not compact-valued relation* $T \circ \varphi$ *instead of* Φ .

To determine some antisymmetric subsets of $S(x)$ ***, it seems also convenient to introduce the following***

Notation 7.2. For any $x \in X$, we define

$$\Gamma_1(x) = \big\{\,(x_1,\ s): \quad s \in [\,x_2,\ -x_1\,]\,\big\}$$

and

$$\Gamma_2(x) = \big\{\,(r,\ x_2): \quad r \in [\,x_1,\ -x_2\,]\,\big\}.$$

Now, as an extension of a former observation that $\{0\}$ ***is a maximal antisymmetric subset of*** $S(0)$ ***, we can also prove the following***

Theorem 7.2. *For any* $x \in X$,

$$\Gamma_1(x) \qquad \text{and} \qquad \Gamma_2(x)$$

are maximal antisymmetric subsets of $S(x)$ *such that*

$$\Gamma_1(x) \cap \Gamma_2(x) = \{x\}.$$

Proof. We shall only prove the corresponding statement for $\Gamma_1(x)$. For this, note that if $s \in [x_2, -x_1]$, then

$$x_2 \leq s \qquad \text{and} \qquad s \leq -x_1.$$

Hence, we can infer that

$$\varphi(x_1, s) = x_1 + s \leq x_1 - x_1 = 0 \quad \text{and} \quad \varphi(x) = x_1 + x_2 \leq x_1 + s = \varphi(x_1, s).$$

Therefore, $(x_1, s) \in X$ and $(x_1, s) \in S(x)$. This shows $\Gamma_1(x) \subseteq S(x)$.

Moreover, if $(x_1, s), (x_1, v) \in \Gamma_1(x)$ such that

$$(x_1, v) \in S(x_1, s) \qquad \text{and} \qquad (x_1, s) \in S(x_1, v),$$

then we can see that

$$\varphi(x_1, s) \leq \varphi(x_1, v) \qquad \text{and} \qquad \varphi(x_1, v) \leq \varphi(x_1, s).$$

Therefore,

$$x_1 + s = \varphi(x_1, s) = \varphi(x_1, v) = x_1 + v,$$

and thus $s = v$. This shows $(x_1, s) = (x_1, v)$, and thus $\Gamma_1(x)$ is an antisymmetric subset of $X(S)$.

Therefore, to complete the proof, we need only show the required maximality property of $\Gamma_1(x)$. For this, suppose that $y \in S(x)$ such that $\Gamma_1(x) \cup \{y\}$ is still an antisymmetric subset of $X(S)$. Then, we can note that

$$\varphi(y) \leq 0 \qquad \text{and} \qquad \varphi(x) \leq \varphi(y).$$

Now, by defining

$$s = \varphi(y) - x_1 \qquad \text{and} \qquad z = (x_1, s),$$

we can note that

$$x_2 = x_1 + x_2 - x_1 = \varphi(x) - x_1 \leq \varphi(y) - x_1 = s \qquad \text{and} \qquad s = \varphi(y) - x_1 \leq -x_1,$$

and thus $z \in \Gamma_1(x)$. Moreover, we can also note that

$$\varphi(z) = x_1 + s = x_1 + \varphi(y) - x_1 = \varphi(y),$$

and thus $\varphi(z) \leq \varphi(y)$ and $\varphi(y) \leq \varphi(y)$. Therefore,

$$y \in S(z) \qquad \text{and} \qquad z \in S(y).$$

Hence, by using the antisymmetry of $\Gamma_1(x) \cup \{y\}$, we can infer that $y = z \in \Gamma_1(x)$. Therefore, $\Gamma_1(x) \cup \{y\} = \Gamma_1(x)$, and thus the required maximality property of $\Gamma_1(x)$ is also true. □

Remark 7.2. *From this theorem, we can see that if* $x \in X$ *such that* $\varphi(x) \neq 0$, *then there does not exist a largest antisymmetric subset of* $S(x)$.

For this, it is enough to note only that

$$u = (x_1, -x_1) \in \Gamma_1(x) \qquad \textit{and} \qquad v = (-x_2, x_2) \in \Gamma_2(x)$$

such that $v \in S(u)$ *and* $u \in S(v)$, *but* $u \neq v$. *Therefore,* $\Gamma_1(x) \cup \Gamma_2(x)$ *is not an antisymmetric subset of* $S(x)$.

8. Monotonicity and Other Mapping Properties

Theorem 8.1. *The following assertions hold:*

(1) φ *is a increasing function of* $X(S)$ *onto* $\mathbb{R}_- =]-\infty, 0]$;

(2) f *is an extensive and intensive function of* $X(S)$ *onto itself such that* f *has no fixed points;*

(3) Φ *is decreasing relation of* $X(S)$ *onto* $\mathbb{R}_-$ *such that* φ *is selection function of* Φ;

(4) S *is a decreasing relation of* $X(S)$ *onto* X *such that* f *is a selection function of* $S \cap S^{-1}$.

Proof. The increasingness of φ with respect the the relations S and $\leq$ was already established in Remark 6.1.

To prove assertion (2), note that, for any $x \in X$, we have $\varphi\big(f(x)\big) = \varphi(x) \leq 0$, and thus $f(x) \in X$. Moreover, we trivially have $\varphi(x) \leq \varphi\big(f(x)\big)$ and $\varphi\big(f(x)\big) \leq \varphi(x)$, and thus

$$f(x) \in S(x) \qquad \text{and} \qquad x \in S\big(f(x)\big).$$

Therefore, f is both extensive and intensive with respect to the relation S.

Hence, we can also see that

$$f(x) \in S(x) \cap S^{-1}(x) = \big(S \cap S^{-1}\big)(x)$$

for all $x \in X$. Therefore, f is also a selection function of $S \cap S^{-1}$.

To prove the decreasingness of the relations S and Φ with respect to the relations S and $\subseteq$, note that by Remark 6.4

$$x\, S\, y \implies S(y) \subseteq S(x),$$

and by the definition of S and Φ

$$x\,S\,y \implies y \in S(x) \implies \varphi(x) \leq \varphi(y) \leq 0 \implies [\varphi(y),\ 0] \subseteq [\varphi(x),\ 0] \implies \Phi(y) \subseteq \Phi(x)$$

for all $x, y \in X$. □

Remark 8.1. *Note that S is actually the largest relation on X which makes the function φ to be increasing with respect to the relations S and $\leq$.*

Namely, if R is such a relation on X, then

$$(x, y) \in R \implies x\,R\,y \implies \varphi(x) \leq \varphi(y) \implies x\,S\,y \implies (x, y) \in S$$

for all $x, y \in X$, and thus $R \subseteq S$.

In addition to Theorem 8.1, it is also worth mentioning the following two facts.

Remark 8.2. *By using the plausible, but ambiguous notation*

$$x^{-1} = (x_1, x_2)^{-1} = (x_2, x_1),$$

we can see that $x_2 + x_1 = \varphi(x) \leq 0$ for all $x \in X$. Therefore,

$$x^{-1} \in X, \qquad \text{and} \qquad \varphi(x^{-1}) = \varphi(x)$$

Thus, X is a symmetric subset *of $\mathbb{R}^2$ and φ is a* symmetric function *of X.*

Remark 8.3. *Hence, by the definition of the relation Φ, we can see that*

$$\Phi(x^{-1}) = [\varphi(x^{-1}),\ 0] = [\varphi(x),\ 0] = \Phi(x)$$

for all $x \in X$. Thus, Φ is symmetric relation *of X onto $\mathbb{R}_-$.*

Moreover, by the definition of S, we can see that

$$y \in S(x)^{-1} \iff y^{-1} \in S(x) \iff \varphi(x) \leq \varphi\big(y^{-1}\big) \leq 0 \iff \varphi(x) \leq \varphi(y) \leq 0 \iff y \in S(x)$$

for all $x, y \in X$. Therefore, $S(x)^{-1} = S(x)$ for all $x \in X$, and thus S is a symmetric-valued relation *of X onto itself.*

However, it is now more important to note that, in addition to Theorem 8.1, we can also prove the following two theorems.

Theorem 8.2. *The following assertions hold:*

(1) $S(0) = \max\big(X\big)$;

(2) *there is no strongly maximal element of $X(S)$;*

(3) *$S(0)$ is the family of all maximal elements of $X(S)$.*

Proof. If $x \in S(0)$, then by Theorem 6.2 we have $x \in \varphi^{-1}(0)$, and thus $\varphi(x) = 0$. Therefore, for any $y \in X$, we have $\varphi(y) \leq 0 = \varphi(x)$, and thus $y \, S \, x$. Hence, we can infer that $x \in \mathrm{ub}(X)$, and thus also $x \in X \cap \mathrm{ub}(X) = \max(X)$. Therefore, $S(0) \subseteq \max(X)$.

Conversely, if $x \in \max(X)$, then we have $x \in \mathrm{ub}(X)$, and thus $y \, S \, x$ for all $y \in X$. Hence, in particular, we can infer that $0 \, S \, x$, and thus $0 = \varphi(0) \leq \varphi(x)$. Moreover, since $x \in X$, we also have $\varphi(x) \leq 0$. Therefore, we actually have $\varphi(x) = 0$. Hence, we can already see that $x \in \varphi^{-1}(0) = S(0)$. Therefore, $\max(X) \subseteq S(0)$, and thus equality (1) also holds.

To prove (2), note that if $x \in X$, then by Theorem 8.1 we have $x \, S \, f(x)$ such that $f(x) \neq x$. Therefore, x cannot be a strongly maximal element of X.

Moreover, since every element of $\max(X)$ is, in particular, a maximal element of $X(S)$, by (1) we can at once see that each element of $S(0)$ is a maximal element of $X(S)$. Therefore, to prove (3), we need only prove the converse statement.

For this, assume that x is a maximal element of $X(S)$. Then, because of $x \in X$, we have $\varphi(x) \leq 0 = \varphi(0)$, and thus $x \, S \, 0$. Hence, by using the maximality of x, we can infer that $0 \, S \, x$, and thus $0 = \varphi(0) \leq \varphi(x)$. Therefore, we actually have $\varphi(x) = 0$, and thus $x \in \varphi^{-1}(0) = S(0)$. □

Theorem 8.3. *The following assertions hold:*

(1) *there is no strong fixed point of* S *;*

(2) *every element of* X *is a fixed point of* S *.*

Proof. By the reflexivity of S, it is clear that (2) is true. Moreover, by Theorem 8.2, we can see that (1) is also true. □

9. Additivity and Homogeneity Properties

Theorem 9.1. *The following assertions hold:*

(1) X *is a pointed convex cone in* $\mathbb{R}^2$ *;*

(2) $S(0)$ *is the largest linear subspace of* $\mathbb{R}^2$ *contained in* X *.*

Proof. If $\lambda \in \mathbb{R}_+$ and $x \in X$, then by the corresponding definitions we have

$$\lambda x = (\lambda x_1, \lambda x_2) \qquad \text{and} \qquad \lambda \geq 0, \quad x_1 + x_2 \leq 0.$$

Hence, we can already infer that

$$(\lambda x)_1 + (\lambda x)_2 = \lambda x_1 + \lambda x_2 = \lambda (x_1 + x_2) \leq 0,$$

and thus $\lambda x \in X$.

Moreover, if $x, y \in X$, then by the corresponding definitions we have

$$x + y = (x_1 + y_1, \, x_2 + y_2) \qquad \text{and} \qquad x_1 + x_2 \leq 0, \quad y_1 + y_2 \leq 0.$$

Hence, we can already infer that

$$(x+y)_1 + (x+y)_2 = x_1 + y_1 + x_2 + y_2 = x_1 + x_2 + y_1 + y_2 \leq 0,$$

and thus $x+y \in X$. Therefore, in particular, assertion (1) is true.

On the other hand, from assertion (4) of Theorem 6.2, we can at once see that $S(0)$ is a linear subspace of $\mathbb{R}^2$. Therefore, to prove (2), we need only prove the stated maximality property of $S(0)$.

For this, assume that Y is a linear subspace of $\mathbb{R}^2$ such that $Y \subseteq X$. Then, for any $y \in Y$, we also have $y \in X$, and thus $\varphi(y) \leq 0$. Moreover, since now $-y \in Y$ also holds, we can also note that $-y \in X$, and thus $\varphi(-y) \leq 0$. However, thus we also have

$$0 \leq -\varphi(y) = -(y_1 + y_2) = -y_1 + (-y_2) = \varphi(-y) \leq 0.$$

Therefore, $\varphi(y) = 0$, and thus by Theorem 6.2 we have $y \in \varphi^{-1}(0) = S(0)$. This shows that $Y \subseteq S(0)$, and thus assertion (2) is also true. □

Remark 9.1. *Recall that if A is a pointed, convex cone in $\mathbb{R}^2$, then we have not only $\lambda A \subseteq A$ for all $\lambda \in \mathbb{R}_+$, but also $\lambda A + \mu A \subseteq A$ for all $\lambda, \mu \in \mathbb{R}_+$.*

Theorem 9.2. *The following assertions hold:*

(1) *φ is a positively homogeneous, additive function of X onto $\mathbb{R}_-$ such that $\varphi[S(0)] = \{0\}$;*

(2) *Φ is a nonegatively homogeneous, additive relation of X onto $\mathbb{R}_-$ such that $\Phi[S(0)] = \{0\}$;*

(3) *S is a positively homogeneous, additive relation of X onto itself such that $S[S(0)] = S(0)$.*

Proof. If $\lambda \in \mathbb{R}_+$, $x \in X$ and $t \in \Phi(x)$, then by using the definition of Φ and assertion (1), we can see that

$$\varphi(x) \leq t \leq 0 \qquad \text{and} \qquad \varphi(\lambda x) = \lambda \varphi(x) \leq \lambda t \leq 0.$$

Therefore, $\lambda t \in \Phi(\lambda x)$, and thus

$$\lambda \Phi(x) \subseteq \Phi(\lambda x).$$

This shows that Φ is nonnegatively superhomogeneous, and thus it is also positively homogeneous.

Quite similarly, if $\lambda \in \mathbb{R}_+$, $x \in X$ and $y \in S(x)$, then by using the definition of S and assertion (1), we can see that

$$\varphi(y) \leq \varphi(y) \qquad \text{and} \qquad \varphi(\lambda x) = \lambda \varphi(x) \leq \lambda \varphi(y) = \varphi(\lambda y).$$

Therefore, $\lambda y \in S(\lambda x)$, and thus

$$\lambda S(x) \subseteq S(\lambda x).$$

This shows that S is also nonnegatively superhomogeneous, and thus it is also positively homogeneous.

In addition to the positive homogeneity of Φ and S, we can also note that

$$\Phi(0x) = \Phi(0) = [\varphi(0),\ 0] = \{0\} = 0\,\Phi(x),$$

but

$$S(0x) = S(0) = \big\{ (r,\ -r): \quad r \in \mathbb{R} \big\} \neq \{0\} = 0\,S(x).$$

Therefore, Φ is 0–homogeneous too, but S is not so even at any point of X.

On the other hand, if $x, y \in X$, $r \in \Phi(x)$ and $s \in \Phi(y)$, then by using the definition of Φ and assertion (1) we can see that

$$\varphi(x) \leq r \leq 0, \quad \varphi(y) \leq r \leq 0 \qquad \text{and} \qquad \varphi(x+y) = \varphi(x) + \varphi(y) \leq r + s.$$

Therefore, $r + s \in \Phi(x+y)$, and thus

$$\Phi(x) + \Phi(y) \subseteq \Phi(x+y).$$

This shows that Φ is superadditive.

Quite similarly, if $x, y \in X$, $z \in S(x)$ and $w \in S(y)$, then by using the definition of S and assertion (1) we can see that

$$\varphi(x) \leq \varphi(z) \qquad \text{and} \qquad \varphi(y) \leq \varphi(w)$$

and

$$\varphi(x+y) = \varphi(x) + \varphi(y) \leq \varphi(z) + \varphi(w) = \varphi(z+w).$$

Therefore, $z + w \in S(x+y)$, and thus

$$S(x) + S(y) \subseteq S(x+y).$$

This shows that S is also superadditive.

Now, it remains actually to show only that both Φ and S are also subadditive. For this, assume first that $x, y \in X$ and $t \in \Phi(x+y)$. Then, by assertion (1) and the definition of Φ, we have

$$\varphi(x) + \varphi(x) = \varphi(x+y) \leq t \leq 0.$$

If $\varphi(x) + \varphi(y) = 0$, then we evidently have $t = 0 + 0 \in \Phi(x) + \Phi(y)$. Therefore, we may assume that $\varphi(x) + \varphi(y) \neq 0$. Define

$$r = \frac{\varphi(x)}{\varphi(x) + \varphi(y)}\, t \qquad \text{and} \qquad s = \frac{\varphi(y)}{\varphi(x) + \varphi(y)}\, t.$$

Then, we evidently have

$$t = \frac{\varphi(x)}{\varphi(x)+\varphi(y)}\, t \;+\; \frac{\varphi(y)}{\varphi(x)+\varphi(y)}\, t \;=\; r+s\,.$$

Moreover, by using the inequalities

$$\varphi(x) \leq 0\,, \quad \varphi(x) \leq 0\,, \qquad \varphi(x)+\varphi(y) < 0\,, \qquad \varphi(x)+\varphi(x) \leq t \leq 0\,,$$

we can see that

$$\varphi(x) \leq \frac{t}{\varphi(x)+\varphi(y)}\, \varphi(x) \;=\; r \;=\; \frac{\varphi(x)}{\varphi(x)+\varphi(y)}\, t \leq 0$$

and

$$\varphi(y) \leq \frac{t}{\varphi(x)+\varphi(y)}\, \varphi(y) = \; s \;=\; \frac{\varphi(y)}{\varphi(x)+\varphi(y)}\, t \leq 0\,.$$

Therefore, $r \in \Phi(x)$ and $s \in \Phi(x)$, and thus $t = r+s \in \Phi(x)+\Phi(y)$ also holds. This shows that

$$\Phi(x+y) \subseteq \Phi(x) + \Phi(y)\,,$$

and thus Φ is subadditive too.

Next suppose that $x, y \in X$ and $\omega \in S(x+y)$. Then, by assertion (1) and the definition of S, we have

$$\varphi(x)+\varphi(y) = \varphi(x+y) \leq \varphi(\omega) \leq 0\,.$$

If $\varphi(x)+\varphi(y) = 0$, then $\varphi(\omega) = 0$, and thus $\omega \in \varphi^{-1}(0) = S(0)$. Hence, by Theorem 6.2, we can see that $\omega = \omega + 0 \in S(x) + S(y)$. Therefore, we may assume that $\varphi(x)+\varphi(y) \neq 0$. Define

$$z = \frac{\varphi(x)}{\varphi(x)+\varphi(y)}\, \omega \qquad \text{and} \qquad w = \frac{\varphi(y)}{\varphi(x)+\varphi(y)}\, \omega\,.$$

Then, we evidently have

$$\omega = \frac{\varphi(x)}{\varphi(x)+\varphi(y)}\, \omega \;+\; \frac{\varphi(y)}{\varphi(x)+\varphi(y)}\, \omega \;=\; z+w\,.$$

Moreover, by using assertion (1) and the inequalities

$$\varphi(x) \leq 0\,, \quad \varphi(y) \leq 0\,, \qquad \varphi(x)+\varphi(y) < 0\,, \qquad \varphi(x)+\varphi(y) \leq \varphi(\omega) \leq 0\,,$$

we can see that

$$\varphi(z) = \frac{\varphi(x)}{\varphi(x)+\varphi(y)}\, \varphi(\omega) \;=\; \frac{\varphi(\omega)}{\varphi(x)+\varphi(y)}\, \varphi(x) \geq \varphi(x)$$

and

$$\varphi(w) = \frac{\varphi(y)}{\varphi(x)+\varphi(y)}\,\varphi(\omega) = \frac{\varphi(\omega)}{\varphi(x)+\varphi(y)}\,\varphi(y) \geq \varphi(y).$$

Therefore, $z \in S(x)$ and $w \in S(y)$, and thus $\omega = z + w \in S(x) + S(y)$. This shows that

$$S(x+y) \subseteq S(x) + S(y),$$

and thus S is subaditive too. □

10. *Translation Properties*

From Theorems 6.1 , 6.2 and 9.2, we can immediately derive the following

Theorem 10.1. *The following assertions hold:*

(1) f *is a translation function;* (2) S *is a translation relation.*

Proof. Namely, by the reflexivity and superadditivity of S, we evidently have

$$x + \mathcal{S}(y) \subseteq S(x) + S(y) \subseteq S(x+y)$$

for all $x, y \in X$. □

Hence, by taking $y = 0$***, we can immediately derive the following***

Corollary 10.1. *For any* $x \in X$*, we have*

$$x + S(0) \subseteq S(x).$$

The following example shows that the corresponding equality need not be true. Therefore, a translation relation even on a pointed convex cone need not be in the expected form [52, Theorem 3.2].

Example 10.1. *If for instance*

$$x = (0, -1) \qquad \textit{and} \qquad y = (-1, 1),$$

then

$$\varphi(x) = x_1 + x_2 = -1 \qquad \textit{and} \qquad \varphi(y) = y_1 + y_2 = 0.$$

and thus $\varphi(x) \leq \varphi(y) \leq 0$. *Therefore,*

$$x, y \in X \qquad \textit{and} \qquad y \in S(x).$$

On the other hand, by using Theorem 6.2, we can see that

$$x + S(0) = (x_1, x_2) + \{ (r, -r) : \quad r \in \mathbb{R} \} \\ = \{ (x_1 + r, x_2 - r) : \quad r \in \mathbb{R} \} = \{ (r, -1 - r) : \quad r \in \mathbb{R} \}.$$

Hence, we can see that $y \notin x + S(0)$, *and thus*

$$S(x) \not\subseteq x + S(0).$$

Namely, if $y \in x + S(0)$, *then by the above equality there exists* $r \in \mathbb{R}$ *such that* $(-1, 1) = y = (r, -1 - r)$. *Therefore,* $-1 = r$, *and thus* $1 = -1 - r = 0$, *which is contradiction.*

Remark 10.1. *By Theorem 6.2 and Remark 7.1, we have*

$$S(0) = \varphi^{-1}(0) \qquad \text{and} \qquad (\varphi^{-1} \circ \varphi)(x) \subseteq S(x)$$

for all $x \in X$.

Therefore, Corollary 10.1 can be improved by proving the following

Theorem 10.2. *For any* $x \in X$, *we have*

$$(\varphi^{-1} \circ \varphi)(x) = x + \varphi^{-1}(0).$$

Proof. If $y \in (\varphi^{-1} \circ \varphi)(x)$, then

$$\varphi(y) = \varphi(x).$$

Define $u = y - x$. Then, $u \in \mathbb{R}^2$ such that $y = x + u$ and

$$u_1 + u_2 = y_1 - x_1 + y_2 - x_2 = y_1 + y_2 - (x_1 + x_2) = \varphi(y) - \varphi(x) = 0.$$

Therefore, $u \in X$ and $\varphi(u) = 0$. Hence, we can already see that $u \in \varphi^{-1}(0)$, and thus

$$y = x + u \in x + \varphi^{-1}(0).$$

This proves that

$$(\varphi^{-1} \circ \varphi)(x) \subseteq x + \varphi^{-1}(0).$$

While, if $y \in x + \varphi^{-1}(0)$, then there exists $u \in \varphi^{-1}(0)$ such that $y = x + u$. Hence, we can infer that $\varphi(u) = 0$. Now, by using Theorem 9.2, we can already see that

$$\varphi(y) = \varphi(x + u) = \varphi(x) + \varphi(u) = \varphi(x),$$

and thus

$$y \in \varphi^{-1}(\varphi(x)) = (\varphi^{-1} \circ \varphi)(x).$$

This proves that

$$x + \varphi^{-1}(0) \subseteq \big(\varphi^{-1} \circ \varphi\big)(x),$$

and thus the required equality is also true. □

Remark 10.2. *Hence, since* $\varphi^{-1}(0) = \varphi\big(\varphi(0)\big) = (\varphi^{-1} \circ \varphi)(0)$, *we can see that*

$$\big(\varphi^{-1} \circ \varphi\big)(x) = x + \big(\varphi^{-1} \circ \varphi\big)(0)$$

for all $x \in X$.

Thus, since $\big(\varphi^{-1} \circ \varphi\big)(0) = \varphi^{-1}(0) = S(0)$ *is a linear subspace of* $\mathbb{R}^2$, $\varphi^{-1} \circ \varphi$ *is also a very particular translation relation on* X.

Moreover, since for any $x, y \in X$ *we have*

$$y \in \big(\varphi^{-1} \circ \varphi\big)(x) \iff \varphi(x) = \varphi(y),$$

it is clear that $\varphi^{-1} \circ \varphi$ *is already an equivalence relation on* X.

From Theorem 10.2, we can easily derive the following

Corollary 10.2. *For any* $r \in \mathbb{R}_-$, *we have*

$$\varphi^{-1}(r) = (r, 0) + \varphi^{-1}(0).$$

Proof. By defining $x = (r, 0)$, we can see that $x \in X$ such that $\varphi(x) = r$. Hence, by using Theorem 10.2, we can already see that

$$\varphi^{-1}(r) = \varphi^{-1}\big(\varphi(x)\big) = \big(\varphi^{-1} \circ \varphi\big)(x)\big) = (r, 0) + \varphi^{-1}(0) = x + \varphi^{-1}(0).$$

□

Remark 10.3. *Thus, by identifying the real number* r *with the complex number* $(r, 0)$, *we can also state that* $\varphi^{-1}(r) = r + \varphi^{-1}(0)$ *for all* $r \in \mathbb{R}_-$.

Now, analogously to Theorem 10.2, we can also prove the following

Theorem 10.3. *For any* $y \in X$, *we have*

$$S^{-1}(y) = y + S^{-1}(0).$$

Proof. If $x \in S^{-1}(y)$, then $y \in S(x)$, and thus

$$\varphi(x) \leq \varphi(y).$$

Define $u = x - y$. Then, $u \in \mathbb{R}^2$ such that $x = y + u$ and

$$u_1 + u_2 = x_1 - y_1 + x_2 - y_2 = x_1 + x_2 - (y_1 + y_2) = \varphi(x) - \varphi(y) \leq 0.$$

Therefore, $u \in X$ and $\varphi(u) \leq 0 = \varphi(0)$. Hence, we can already see that $0 \in S(u)$, and thus $u \in S^{-1}(0)$. Therefore,

$$x = y + u \in y + S^{-1}(0).$$

This proves that

$$S^{-1}(y) \subseteq y + S^{-1}(0).$$

While, if $x \in y + S^{-1}(0)$, then there exists $u \in S^{-1}(0)$ such that $x = y + u$. Now, we can note that

$$y \in S(y) \qquad \text{and} \qquad 0 \in S(u).$$

Hence,by using the superadditivity of S we can infer that

$$y = y + 0 \in S(y) + S(u) \subseteq S(y + u) = S(x),$$

and thus $x \in S^{-1}(y)$ also holds. This proves that

$$y + S^{-1}(0) \subseteq S^{-1}(y),$$

and thus the required equality is also true. □

From this theorem, by using Remark 6.6, we can immediately derive

Corollary 10.3. *For any $y \in X$, we have*

$$S^{-1}(y) = y + X.$$

Remark 10.4. *Hence, since X is a pointed convex cone in $\mathbb{R}^2$, we can see that S^{-1} is also a very particular translation relation on X.*

11. Possibilities for Some Further Investigations

In addition to the algebraic properties of the relations considered in this paper, some topological properties of these relations can also be investigated.

For this, note that on the sets X and $\mathbb{R}$ we may naturally consider the singleton relators $\{S\}$ and $\{\leq\}$, respectively.

Moreover, on the set $\mathbb{R}$ we may also naturally consider the countable relator $\mathcal{R} = \{R_n\}_{n \in \mathbb{N}}$, with

$$R_n = \{(r, s) \in \mathbb{R}^2 : \quad d(r, s) < 2^{-n}\}.$$

Furthermore, on the set $\mathbb{R}^2$ we may also naturally consider more than one similar singleton and countable relators.

Namely, on $\mathbb{R}^2$, in addition to the usual coordinate-wise partial order, we may also naturally consider the lexicographic order.

Moreover, on $\mathbb{R}^2$, instead of the usual Euclidean metric, we may also consider the postman, radial and river metrics [61].

However, it is now more important to note that some of the assumptions of Notation 6.1, and the results of this paper, can be modified and generalized.

First of all, instead of the preference relation S, we may also naturally consider the Brondsted relation

$$S_d = \{ (x, y) \in X^2 : \quad d(x, y) \leq \varphi(y) - \varphi(x) \}$$

with a quasi-pseudo-metric d on $\mathbb{R}^2$. Namely, thus we have $S = S_0$.

Moreover, we may naturally look for some sufficient conditions on d in order that some of the theorems on S could remain true for S_d.

On the other hand, it would also be of some interest to find some counterexamples to Fierro's Theorem 1.1 by using Pervin and Cantor relations instead of S.

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